

On locally optimal tests for the mean direction of the Langevin distribution

A. SenGupta

Indian Statistical Institute, Calcutta, WB 700035, India, and Department of Statistics and Applied Probability, University of California, Santa Barbara, CA 93106, USA

S. Rao Jammalamadaka

Department of Statistics and Applied Probability, University of California, Santa Barbara, CA 93106, USA

Abstract: Hayakawa (1990) has very recently studied the behavior of the power for several large sample tests for the mean direction vector of the Langevin distribution. These tests are not known to possess any non-trivial optimal property. Here we derive some multiparameter locally optimal tests, e.g., best first and second directional derivative tests and locally most mean powerful tests. For the case when κ , the concentration parameter, is known, these tests are exact and some cut-off points are presented. For the case when κ is unknown, we propose $C(\alpha)$ -type tests which are expected to be asymptotically locally optimal. Some open problems are indicated.

1. Introduction

For testing the mean direction vector of the Langevin distribution $L(\mu, \kappa)$, Hayakawa (1990), has compared the Likelihood ratio test, Rao's test, Watson's test and modified Wald's test, using asymptotic expansions and simulations. In general, no small sample optimal properties of these tests could be claimed and as expected none of them shows uniform dominance. Also they are known to be asymptotically equivalent. These large sample tests, usually employed as the last resorts, have been widely criticized.

The least one should require of a test, is that it be admissible. Although the Langevin distribution is a member of the regular exponential family, difficulties arise when testing for μ . Even with κ known, the distribution reduces to a curved exponential family and general approaches e.g. Bayes test (Kiefer and Schwartz, 1965), Stein's (1955) generalization of Birnbaum's theorem etc. to obtain admissible tests for the exponential family are thus either not applicable or difficult to compute. This has made the locally best or locally most

powerful (LMP) test (see e.g. Ferguson, 1967) a popular choice in the one-parameter case. Based on the criterion of statistical curvature (Amari, 1985; Efron, 1975), one can even initially obtain an idea of the performance of the LMP test and the sample size required to expect it to be reasonable. A unique LMP test is, of course, admissible.

However, for the multiparameter case, there are several versions of local bestness and hence several types of LMP tests. In this note, we take up the problem dealt by Hayakawa. But, in contrast to his consideration of the large sample tests whose (non-trivial) optimality properties are unknown, the aim here is to derive tests which are locally optimal. We also point out the difficulties that one encounters in fulfilling this aim and suggest some open problems for future research. Some numerical results through simulations, are also presented.

We first consider tests which maximize the power in a preferred direction (John, 1971). Since maximizing power in any given direction will usually result in pulling down powers in other directions, we consider next maximizing 'overall' power. Type D or locally most total power unbiased

(LMTPU) test proposed by Isaacson (1951) maximizes total curvature of the power hypersurface. However, the critical region has to be guessed and is thus practically inconvenient. Well known optimal tests may not be optimal in the sense of total power as e.g. Hotelling's T^2 - and R^2 -tests (Giri and Kiefer, 1964). Finally LMTPU test may not even exist (e.g. SenGupta and Vermeire, 1986, henceforth SV) as for instance, for the important problem of testing $\mu = 0, \sigma^2 = 1$ for a normal population. Locally most mean powerful unbiased (LMMPU) test proposed by SV maximizes mean curvature of the power hypersurface, is easy to construct and coincides with several popular tests in standard problem, e.g., Hotelling's T^2 , a test in linear models (Scheffe, 1959) etc. We derive the LMMPU test. A similar approach due to Cohen et al. (1985) will not be discussed here. It should be pointed out that for both the directional derivative and the LMMPU test, requiring unbiasedness poses computational problems. For the former, the test depends on the direction chosen inextricably. For the latter, we show elsewhere that the LMP test is globally unbiased in the one-parameter case (SenGupta and Jammalamadaka, 1989, henceforth SJ). We conjecture that for the LMMPU test also (because of the symmetry of the Langevin distribution), the coefficients corresponding to the local unbiasedness conditions can be taken to be zeros through approaches similar to those for the Cauchy or normal distributions or through variations of Anderson's theorem (Tong, 1980). Thus the LMMPU test will result in a very simple criterion.

In case κ is unknown, the problem is open. In this case there is no useful exact invariant or unconditional similar tests. Thus standard methods for obtaining unconditional LMMPU similar (e.g. SV), LMMPU Invariant, locally minimax (e.g. Giri and Kiefer, 1964) or locally best invariant (e.g. Kariya, 1978) tests through Wijsman's representation of the distribution of the maximal invariant and application of the Hunt-Stein theorem, are useless. For the one-parameter case, as an unconditional optimal test, the asymptotically LMP test has been derived in SJ as the Neyman's $C(\alpha)$ test (Neyman, 1959), since all the conditions are satisfied. There is quite some interesting literature (e.g. Chibisov, 1978) also on the asymptotic distribution of the $C(\alpha)$ test statistic. However,

for the multiparameter case, we are unaware of any generalization of the $C(\alpha)$ test. For the κ unknown case, we thus propose, recalling our result for the one-parameter case, replacing the locally optimal tests for the κ known case by $\hat{\kappa}_0$, the MLE of κ under H_0 .

Clearly local optimality does not guarantee good global properties like monotonicity etc. and even consistency, which need to be investigated separately. These global properties have been established for the one-parameter case by us in SJ and one would expect them to hold good for the multiparameter case also. Further, since good local behavior is often practically essential and theoretically necessary for good global behavior, locally optimal tests are definitely reasonable candidates for multiparameters tests of hypothesis in general, and for testing the mean direction vector of the Langevin distribution, in particular.

2. Locally best test

A p -dimensional unit random vector X follows the Langevin or the p -variate von Mises-Fisher distribution $L_p(\mu, \kappa)$ with mean (modal) direction vector μ and concentration parameter κ if its density function on the surface of a unit hypersphere S_p , centered at the origin, is given by

$$\begin{aligned}
 f(x; \mu, \kappa) &= a_p^{-1}(\kappa) \exp(\kappa x' \mu), & (2.1) \\
 \kappa &> 0, \quad \mu' \mu = 1, \\
 x \in S_p &= \{x \mid x \in \mathbb{R}^p, x' x = \|x\|^2 = 1\}, \\
 a_p(\kappa) &= (2\pi)^{p/2} I_{p/2-1}(\kappa) \cdot \kappa^{1-p/2},
 \end{aligned}$$

and $I_\nu(\kappa)$ being the modified Bessel function of the first kind of order ν . Consider the polar transformation $y = u(\gamma)$, y and γ being p - and $(p-1)$ -dimensional vectors respectively. Then, letting θ and α be the spherical polar coordinates of x and μ respectively, the p.d.f. of θ from (2.1) can be written as

$$\begin{aligned}
 g(\theta, \alpha, \kappa) &= a_p^{-1}(\kappa) [\exp\{\kappa u'(\theta) u(\alpha)\}] b_p(\theta), \\
 0 \leq \theta_i \leq \pi, \quad i &= 1, \dots, p-2, \quad 0 \leq \theta_{p-1} < 2\pi, \\
 dS_p &= b_p(\theta) d\theta, \quad b_p(\theta) = \prod_{j=2}^{p-1} \sin^{p-j} \theta_{j-1}, \\
 b_2(\theta) &= 1. & (2.2)
 \end{aligned}$$

For $p = 2$, the distribution is termed the von Mises or circular normal distribution and reduces to

$$g_1(\theta; \alpha, \kappa) = \{2\pi I_0(\kappa)\}^{-1} \exp\{\kappa \cos(\theta - \alpha)\},$$

$$0 \leq \theta < 2\pi, \tag{2.3}$$

and $y_1 = u_1(\gamma) = \sin \gamma$, $y_2 = u_2(\gamma) = \cos \gamma$.

For $p = 3$, the distribution is termed the Fisher distribution and reduces to

$$g_2(\theta_1, \theta_2; \alpha_1, \alpha_2, \kappa)$$

$$= \{\kappa/(4\pi \sin \kappa)\} \exp[\kappa\{\cos \alpha_1 \cos \theta_1$$

$$+ \sin \alpha_1 \sin \theta_1 \cos(\theta_2 - \alpha_2)\}] \sin \theta_1,$$

$$0 \leq \theta_1 \leq \pi, \quad 0 \leq \theta_2 < 2\pi, \tag{2.4}$$

and $y_i = u_i(\gamma_1, \gamma_2)$, $i = 1, 2, 3$, $y_1 = \sin \gamma_1 \sin \gamma_2$, $y_2 = \sin \gamma_1 \cos \gamma_2$, $y_3 = \cos \gamma$. For a random sample X_1, \dots, X_n of size n from the Fisher distribution, the p.d.f. of the sufficient statistic $R = (R_1, R_2, R_3)'$, $R_i = n\bar{X}_i$, $i = 1, 2, 3$ under $\alpha = \mathbf{0}$ is given by

$$h_\kappa(R) = \{2c(\kappa)\}^n h_0(R) \exp(\kappa R_3). \tag{2.5}$$

The p.d.f. of R_3 is given by

$$f(R_3) = \{c(\kappa)\}^n \phi_{n,n-1}(R_3) \exp(\kappa R_3),$$

$$|R_3| < n. \tag{2.6}$$

Further, under the polar transformation, letting $R^2 = \sum_{i=1}^3 R_i^2$, the p.d.f. of R given R_3 is

$$f(R | R_3; \kappa) = \phi_n(R) / \phi_{n,n-1}(R_3),$$

$$|R_3| < R < n, \quad \phi_n(R) = \phi_{n,n-2}(R). \tag{2.7}$$

The expressions for $c(\kappa)$, h_0 and $\phi_{i,j}$ are available from Mardia (1972). In practice we will be concerned mostly with $p \leq 3$ and the above expressions will be found useful to give exact results. For general p , (2.1) being a multivariate generalization of (2.3) and (2.4), is often termed the p -variate von Mises–Fisher distribution. When μ and κ are unknown, (2.1) is a member of the p -dimensional regular exponential family with the sufficient statistics \bar{X} for the parameter vector $(\mu_1, \dots, \mu_{p-1}, \kappa)'$. However, when κ is known, say $\kappa = \kappa_0$, this distribution becomes a member of the curved exponential family (CEF), a $(p-1, p)$ CEF (in the notation of Amari, 1985) — im-

bedded in the p -dimensional regular exponential family

$$g(x; \mu, \kappa_0)$$

$$= C(\kappa_0, \mu) \exp\{-\frac{1}{2}\kappa_0(x - \mu)'(x - \mu)\},$$

$$x \text{ and } \mu \in p[-1, 1]^p,$$

with the p -dimensional sufficient statistic \bar{X} for the $(p-1)$ -dimensional parameter vector $(\mu_1, \dots, \mu_{p-1})'$ or equivalently for $\alpha = (\alpha_1, \dots, \alpha_{p-1})'$.

It is clear from (2.3) that α is a location parameter. To identify μ in (2.1) as a location parameter, note simply that $\mu'x = 1 - \frac{1}{2}(x - \mu)'(x - \mu)$ since $x'x = \mu'\mu = 1$. Hence results pertaining to a location parameter vector may be exploited if necessary.

We are interested in tests for the parameter α . Note from (2.3) that without loss of generality we can test for $H_0: \alpha = \alpha_0 \equiv \mathbf{0}$, i.e., for $H_0: \mu = \mu_0 \equiv (0, 1)'$. For the multiparameter case, $H_0: \alpha = \alpha_0 \equiv \mathbf{0}$ corresponds to $H_0: \mu = \mu_0 \equiv (0, \dots, 0, 1)'$. Thus $H_0: \mu = (0, \dots, 0, 1)'$ has received considerable attention (Mardia, 1972) and large sample tests, e.g. likelihood ratio test, have been investigated. In fact, since μ is the location parameter, in testing for the mean direction one may without loss of generality, confine to this specific μ_0 .

Case 1. κ known

Note the CEF representation of (2.1) for known κ . Rather than searching for an ancillary statistics (exact, local, second order etc.) and using a conditional test based on the principle of ancillarity by Fisher, we propose here an unconditional test. Unfortunately the distribution of the sufficient statistic \bar{X} is not available in a closed form and we will need to deal with the likelihood function using the original observations. Also let $\kappa = 1$. We will present unconditional small sample tests which are locally best (most powerful) according to different notions or criteria.

(a) *Maximum directional power test.* We first consider a test φ that is best for detecting small deviations from H_0 . It is derived by maximizing power in a preferred direction, say η_0 , through the use of the generalized Neyman–Pearson lemma. This test has the optimal property of being a

locally best test in the sense that for every other test $\tilde{\varphi}$ of the same size, there is a neighborhood $N(\eta_0)$ of η_0 such that the power of φ is not smaller than that of $\tilde{\varphi}$. Further $\tilde{\varphi}$ maximizes the rate of change of the power function in the direction η_0 . A similar approach has been used by John (1971) to obtain locally optimal tests for covariance matrices of multivariate normal populations.

The first directional derivative of $\ln f(\mathbf{x}; \boldsymbol{\mu}, 1)$ with respect to μ_i 's, $i = 1, \dots, p - 1$ in the direction $\boldsymbol{\eta} = (\eta_1, \dots, \eta_{p-1})'$ at $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, $\boldsymbol{\mu}'_0 \boldsymbol{\mu}_0 = 1$, is given by

$$\sum_1^{p-1} \left(x_i + x_p \frac{\partial \mu_p}{\partial \mu_i} \right) \eta_i \Big|_{\boldsymbol{\mu}_0} = \sum_1^{p-1} x_i \eta_i - (x_p / \mu_{p0}) \left(\sum_1^{p-1} \eta_i \mu_{i0} \right)$$

where we recall that $\mu_p^2 = 1 - \sum_1^{p-1} \mu_i^2$. Then, for a random sample $\mathbf{x}_1, \dots, \mathbf{x}_n$ of size n , by the generalized Neyman-Pearson lemma, the test that maximizes the corresponding first directional derivative of the power function has the critical region

$$\omega: T = \sum_1^{p-1} \bar{x}_i \eta_i - (\bar{x}_p / \mu_{p0}) \left(\sum_1^{p-1} \eta_i \mu_{i0} \right) > C \tag{2.8}$$

where C is chosen to give the desired level of significance.

Consider for example $\boldsymbol{\eta}_0 = a \mathbf{1}$, $a = \pm 1/\sqrt{p-1}$. Further, as a special case, let $\boldsymbol{\mu}_{01} = (0, \dots, 0, 1)'$. Then the critical region reduces to

$$\omega: T_i = \sum_1^{p-1} \bar{x}_i > (<) C \text{ for } a > (<) 0.$$

As another special case, let $\boldsymbol{\mu}_{02} = (1/\sqrt{p}) \times (1, \dots, 1)'$. Then, we have the critical region as

$$\omega: T_2 = \sum_1^{p-1} \bar{x}_i - \bar{x}_p > (<) K \text{ for } a > (<) 0.$$

For $p = 2$, $\{\boldsymbol{\mu}_0 = (0, 1)'\} \equiv \{\alpha_0 = 0\}$ and the test for $H_0: \alpha_0 = 0$ against $H_1: \alpha > 0$ then has the critical region

$$\omega: T_1 = \left(\sum_1^n \sin \theta_j \right) / n > C$$

which agrees with that previously obtained by SJ. Also,

$$\begin{aligned} T_2 &= \bar{x}_1 - \bar{x}_2 \\ &= \left(\sum_1^n \sin \theta_j - \sum_1^n \cos \theta_j \right) / n \\ &= (S - C) / n \\ &= \bar{S} - \bar{C}. \end{aligned}$$

The joint distribution of S and C are available from (4.5.4) in Mardia (1972) from which the p.d.f. of T_2 may be obtained. For $p = 3$, $T_1 = \bar{R}_1 + \bar{R}_2$ and $T_2 = \bar{R}_1 + \bar{R}_2 - \bar{R}_3$ where $\bar{R}_i = R_i/n$, $i = 1, 2, 3$. The p.d.f.'s of T_1 and T_2 may be obtained from (2.5).

For other directions $\boldsymbol{\eta}$ and $\boldsymbol{\mu}_0$ in general, the critical region will depend more intricately on these vectors chosen. These tests remind us of the one-sided LMP test in the one-parameter case and as in there, here also unbiasedness, either local or global, for the multiparameter case needs to be verified.

Recalling the two-sided locally unbiased LMP-test for the one-parameter case, we may attempt as above to obtain a test that maximizes the second directional derivative among tests for which the first directional derivative is zero. This may be looked upon as maximizing 'directional curvature' after invoking the condition of 'directional unbiasedness' in the direction $\boldsymbol{\eta}$ at $\boldsymbol{\mu}_0$. This test, however, is not unbiased or even LU. The conditions for a test to be LU in the multiparameter case are given below with the discussions on the LMMPU test. The critical region is given by

$$\begin{aligned} \omega: \tilde{T} &\equiv n^2 T^2 \\ &- n \frac{\bar{x}_p}{\mu_{p0}} \left\{ \sum_1^{p-1} \eta_i^2 + \frac{1}{\mu_{p0}^2} \left(\sum_1^{p-1} \eta_i \mu_{i0} \right)^2 \right\} \\ &- C - C_1 T \geq 0, \end{aligned}$$

where T is defined in (2.8) and C and C_1 are constants to be so chosen as to satisfy the conditions of size and directional unbiasedness. It is easily seen that this test depends on the direction inextricably for a general $\boldsymbol{\mu}_0$. However, for $\boldsymbol{\mu}_{01}$

and μ_{02} it is given respectively by the simpler forms

$$\omega: \tilde{T}_1 \equiv n^2 \left[\sum_1^{p-1} \bar{x}_i \eta_i + A_1 \right]^2 - n\bar{x}_p \left(\sum_1^{p-1} \eta_i^2 \right) + B_1 \geq 0$$

and

$$\omega: \tilde{T}_2 \equiv n^2 \left[\sum_1^{p-1} (\bar{x}_1 - \bar{x}_p) \eta_i + A_2 \right]^2 - \bar{x}_p n \sqrt{p} \left\{ \sum_1^{p-1} \eta_i + \left(\sum_1^{p-1} \eta_i \right)^2 \right\} + B_2 \geq 0.$$

For $p = 2$ and $p = 3$ one may attempt to derive the exact distributions for \tilde{T} , \tilde{T}_1 , and \tilde{T}_2 using (4.5.4) of Mardia and (2.5) respectively.

A two-sided size- α test that is locally directional unbiased for any direction can also be derived easily following John (1971). For example, the acceptance region $\bar{\omega}$ can be given by, $\bar{\omega}_i: T_i \in [c_{i1}, c_{i2}]$, T_i 's corresponding to μ_{0i} , $i = 1, 2$, defined as above, and where c_{i1} and c_{i2} are chosen so that

$$(1) \int_{c_{i1}}^{c_{i2}} g_0(t_i) dt_i = 1 - \alpha$$

and

$$(2) \int_{c_{i1}}^{c_{i2}} t_i g_0(t_i) dt_i = 0,$$

where $g_0(t_i)$ is the null density of T_i under μ_{0i} , $i = 1, 2$, and $\kappa = 1$. These tests, however, do not maximize the second directional directional derivative of the power function and we are not aware of any optimal property of such tests. It would be interesting to derive a test which is optimal in the sense that it maximizes principal (power) curvature corresponding to some principal (power) direction of the (test φ) power hypersurface β at μ_0 , rather than considering any arbitrary direction η . This is left for future research.

Note the elegant forms for T_1 and T_2 and even T . For $p = 3$, Table 1 gives the cut-off points for the first directional derivative test based on T_1 and T_2 with $a > 0$. 10 000 values of each T_i , $i = 1, 2$, were generated corresponding to each combination of (κ, n, α) , $\kappa = 1, 5, 10$; $n = 10, 15, 20$ and $\alpha = 0.01, 0.05, 0.10$. Though the exact distributions seem quite complicated, the asymptotic distributions, however, can be easily given from which the cut-off points for large values of n may then be obtained. By the central limit theorem, for a random sample X_i , $i = 1, \dots, n$, for large n , $\sqrt{n}(\bar{X} - \theta) \approx N_p(\mathbf{0}, \Omega)$ where θ , Ω and more refined approximations are given in Hayakawa (1990). Then for large n , the null and non-null distributions of T and T_i are easily obtained. For example, $\sqrt{n} T_i \approx N(\mathbf{a}'_i \theta, \mathbf{a}'_i \Omega \mathbf{a}_i)$, $i = 1, 2$, where $\mathbf{a}_1 = (1, \dots, 1, 0)'$ and $\mathbf{a}_2 = (1, \dots, 1, -1)'$ respectively. For large n , the null and non-null distributions of \tilde{T} and \tilde{T}_i , $i = 1, 2$, can also be easily derived by use of the Delta method.

Table 1
Cut-off points of T_1 - and T_2 -tests for certain values of κ , n and α

	α	$n = 10$		$n = 15$		$n = 20$	
		T_1	T_2	T_1	T_2	T_1	T_2
$\kappa = 1$	0.10	0.3422	0.0942	0.2550	-0.0021	0.2442	-0.0244
	0.05	0.4447	0.2005	0.3225	0.0950	0.3015	0.0456
	0.01	0.6279	0.4173	0.4671	0.2435	0.4143	0.2506
$\kappa = 5$	0.10	0.2232	-0.5547	0.1860	-0.6081	0.1629	-0.6248
	0.05	0.2796	-0.4790	0.2410	-0.5353	0.2057	-0.5705
	0.01	0.4106	-0.3296	0.3259	-0.4217	0.2882	-0.4850
$\kappa = 10$	0.10	0.1580	-0.7315	0.1325	-0.7596	0.1298	-0.7591
	0.05	0.2014	-0.6847	0.1694	-0.7174	0.1688	-0.7214
	0.01	0.3111	-0.5803	0.2536	-0.6371	0.2149	-0.6697

(b) *LMMPU and LMTPU test.* We present below some basic notions and results for the LMMPU and LMTPU tests which are two-sided LU tests. For more details, the reader is referred to SV.

Consider testing $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$. A level α test φ is (strictly) LU if $\beta_\varphi(\mu_0) (>) \geq \alpha$ for all μ in some neighborhood of μ_0 . A test φ with $\beta(\mu)$ in C^2 strictly LU of level α iff (1) $\beta(\mu_0) = \alpha$, (2) $\dot{\beta}(\mu_0) = 0$, and (3) $\ddot{\beta}(\mu_0)$ is positive definite. The level α test φ is LMMPU if it is LU and for any other LU level α test ψ , \exists and $r_0 > 0$ (depending on ψ) such that

$$\int_{S_r} \beta_\varphi(\mu) d\mu > \int_{S_r} \beta_\psi(\mu) d\mu,$$

$$S_r = \{ \mu: |\mu - \mu_0| < r \}, \quad r < r_0.$$

Let $H = \text{tr } \ddot{\beta}(\mu_0)$ and $K = \det \ddot{\beta}(\mu_0)$. A test- φ is regular LMMPU/LMTPU iff $\beta(\mu)$ satisfies (1)–(3) and (4) $H_\varphi \geq H_\psi$ / (4)' $K_\varphi \geq K_\psi$ for any other level α test ψ . LMMPU test maximizes the mean curvature H while LMTPU test maximizes the total or Gaussian curvature K . A critical region for a LMTPU test needs to be guessed and may not exist. We leave this for future research for our problem. The critical region for LMMPU test, however, can easily constructed using the result below, given as Theorem 2 in SV.

Result 1. Let $f(x, \theta)$, $\theta \in \Omega \subset \mathbb{R}^k$, be a k -parameter family of densities, $\in \mathbb{R}^n$. Let $H_0: \theta = \theta_0$ be a null hypothesis. Assume that the integral and derivative can be interchanged in β . Consider any Borel set ω of the form

$$\omega: \sum_1^k \ddot{f}_{ii}(x, \theta_0) \geq c f(x, \theta_0) + \sum_1^k c_i \dot{f}_i(x, \theta_0), \tag{3.1}$$

where the constants c, c_1, \dots, c_k satisfy the conditions

$$(1) \int_\omega f(x, \theta_0) = \alpha$$

and

$$(2) \int_\omega \dot{f}_i(x, \theta_0) = 0 \quad (1 \leq i \leq k).$$

Moreover, if ω is essentially the only set with the property

$$(3) \quad \ddot{\beta}(\theta_0) = \left(\int_\omega \ddot{f}_{ij}(x, \theta_0) \right) \in \mathbb{R}^{k \times k}$$

is positive definite, then ω is regular LMMPU level α critical region. If $\ddot{\beta}(\theta_0)$ is positive semidefinite, one has obtained a second-order LMMPU level α critical region. \square

Expressed in the log-likelihood, $l(x, \theta) \equiv \ln f(x, \theta)$, we have

$$\begin{aligned} \omega: \quad & \sum_1^k [\ddot{l}_{ii}(x, \theta_0) + \dot{l}_i^2(x, \theta_0)] \\ & \geq c + \sum_1^k c_i \dot{l}_i(x, \theta_0). \end{aligned} \tag{3.2}$$

Application of Result 1, on assuming $\kappa = 1$ and recalling $\mu_p^2 = 1 - \sum_1^{p-1} \mu_i^2$, gives

$$\begin{aligned} \omega: \quad & V_1 = n^2 \sum_1^{p-1} u_i^2 - n \frac{\bar{x}_p}{\mu_{p0}} \left\{ p - 1 + \sum_1^{p-1} \left(\frac{\mu_{i0}}{\mu_{p0}} \right)^2 \right\} \\ & \geq c + \sum_1^{p-1} c_i u_i \end{aligned} \tag{3.3}$$

where $u_i = \mu_{i0} \{ (\bar{x}_i / \mu_{i0}) - (\bar{x}_p / \mu_{p0}) \}$, and c and $c_i, i = 1, \dots, p - 1$, are so chosen as to satisfy the following conditions:

- (1) $\int_\omega f_0 = \alpha$,
- (2) $\int_\omega \left\{ \frac{\bar{x}_i}{\mu_{i0}} - \frac{\bar{x}_p}{\mu_{p0}} \right\} f_0 = 0, \quad i = 1, \dots, p - 1$,
- (3) $(\ddot{\beta}_{i,j;0})$ is positive definite

where

$$\ddot{\beta}_{i,j;0} = -n\kappa_0 \frac{\mu_{i0}}{\mu_{p0}} \int_\omega \bar{x}_p f_0, \quad i \neq j,$$

and

$$\begin{aligned} \ddot{\beta}_{i,i;0} = \int_\omega \left[\left\{ \frac{n\kappa_0}{\mu_{i0}} \left(\frac{\bar{x}_i}{\mu_{i0}} - \frac{\bar{x}_p}{\mu_{p0}} \right) \right\}^2 \right. \\ \left. - n\kappa_0 \frac{\bar{x}_p}{\mu_{p0}} \left\{ 1 + \left(\frac{\mu_{i0}}{\mu_{p0}} \right)^2 \right\} \right] f_0, \end{aligned}$$

$$i, j = 1, \dots, p - 1,$$

and where $f_0 \equiv f(x; \mu_0, \kappa_0)$ defined in (2.1) being the p.d.f. of X under $H_0: \mu = \mu_0$ with $\kappa = \kappa_0$ known.

For the case $\mu_0 \equiv \mu_{01} = (0, \dots, 0, 1)'$, $\kappa_0 = 1$, the above conditions reduce to

$$(1) \int_{\omega} f_0 = \alpha,$$

$$(2)' \int_{\omega} \bar{x}_i f_0 = 0$$

and

$$(3)' \int_{\omega} \{ (n\bar{x}_i)^2 - n\bar{x}_p \} f_0 \geq 0, \quad i = 1, \dots, p - 1.$$

Note that since in this case, $\beta_{i,j,0} = 0$, (3) greatly simplifies and reduces to (3)'.

In many problems, e.g. in univariate and multivariate normal distributions considered in SV, c_i 's of (3.2) can be chosen to be zeros by exploiting the symmetry of the distribution. (See also Ferguson, 1967, p. 239.) It is also well-known that this is true for large samples in the one-parameter case under very general regularity conditions. For our case, let $c_i = 0$, $i = 1, \dots, p - 1$. Then (3.3) reduces to

$$\omega: \sum_1^{p-1} \bar{x}_i^2 - np \left(\bar{R}^2 - \sum_1^{p-1} \bar{x}_i^2 \right)^{1/2} \geq c \quad (3.4)$$

i.e.,

$$\omega: V_2 = (\bar{R}^2 - \bar{x}_p^2) - np\bar{x}_p \geq c \quad (3.5)$$

where $\bar{R}^2 \equiv (R/n)^2 = \sum_i^p \bar{x}_i^2$ is obtained through the polar transformation and c is chosen to meet the desired level α .

We recall from Hayakawa (1990) that for large n , by virtue of the central limit theorem, $\sqrt{n}(\bar{X} - A\mu) \sim N_p(0, \Omega)$ where $A = d \ln a_p(\kappa) / d\kappa$ and $\Omega = A' \mu \mu' + A(I - \mu \mu') / \kappa$, $A' = dA / d\kappa$. Thus \bar{X} and also \bar{x}_p have asymptotically symmetric (normal distributions). Further, V_2 is an even function of \bar{x}_i , $i = 1, \dots, p - 1$. Hence, it is expected that asymptotically the critical region in (3.5) will satisfy the conditions of null gradient vector and positive definite Hessian matrix for the power at μ_0 . Generalization of the approach using Anderson's theorem (Tong, 1980) as exploited in SV for

the LMMPU test for the mean vector of the multivariate normal distribution may be helpful. Note, however, that both $E(\bar{X})$ and $\text{Disp}(\bar{X})$ depend on μ . Certainly ω in (3.4) has a very elegant form and the distribution of V for large samples can be easily obtained through the Delta method. We conjecture that (3.5) will give a LM-MPU test.

In general, the cut-off points for V have to be obtained through simulation since its distribution is quite complicated. However, for all practical purposes, $p \leq 3$ and in these cases the distribution may be obtained analytically. For $p = 3$, let $W = n^2 V_2 = (R^2 - R_3^2) - 3n^2 R_3$. Then,

$$\left\{ \begin{aligned} F_W(w) &= \int_u [G(t) - G(-t)] f(u) du, \\ t &= u^2 + 3n^2 u + w \end{aligned} \right\}$$

where $G(t) \equiv F(R | R_3 = u)$ and $f(R_3)$ are available from (2.7) and (2.6) respectively. One may, in this case, thus obtain the cut-off points through numerical integration.

Case 2. κ unknown

Let κ be unknown. Then the principle of similarity, even for $p = 2$ leads to a conditional test. There is also no useful invariance with respect to κ . The aim here is to obtain unconditional optimal tests. For $p = 2$ it has been shown in SJ that the conditions for Neyman's $C(\alpha)$ test to be valid are all satisfied with $\kappa \leq K < \infty$. A $C(\alpha)$ test is asymptotically LMP for the one-parameter case. Imitating the statistic for the one-parameter case, we propose $C(\alpha)$ -type tests for the multiparameter situation. We hope that these tests will also be asymptotically optimal-optimality corresponding to the κ known case — asymptotically best first/second directional derivative test, asymptotically LMMPU test, etc. Let $S(\kappa)$ be a test statistic when κ is known. Under $H_0: \mu = \mu_0$, when κ is unknown, let $\hat{\kappa}_0$ be a locally root n consistent estimator of κ . For example, the maximum likelihood estimator is easily seen to be such an estimator. Define

$$\hat{S} = [S(\hat{\kappa}_0) - \hat{E}_0\{S(\kappa)\}] / [\hat{\text{Var}}_0\{S(\kappa)\}]^{1/2}.$$

For the κ unknown case, we propose the test statistics \hat{T} , \hat{T}_i , \tilde{T} , \tilde{T}_i and \hat{V}_i , $i = 1, 2$, remembering to replace by $\hat{\kappa}_0$ wherever we had used $\kappa = 1$ previously. For \hat{T} and \hat{T}_i , $i = 1, 2$, $E_0 \equiv 0 \equiv \hat{E}_0$. Hayakawa (1990) has shown that $E(\bar{X}) = A\mu$ and $\text{Var}(\bar{X}) = \Omega$ for all n . So E_0 and Var_0 for \hat{T} and \hat{T}_i are easily obtained for all sample sizes, while those for the others may be obtained for large samples by use of the asymptotic normality of \bar{X} . The exact distributions of these test statistics are complicated and cut-off points have to be obtained through simulation or by the method of bootstrap. Asymptotically, however, these will all be normally distributed. Investigations in these directions are left for future research.

References

- Amari, S. (1985), Differential-geometrical methods in statistics, in: D. Brillinger et al., eds., *Lecture Notes in Statist. No. 28* (Springer, Berlin).
- Chibisov, D.M. (1978), An asymptotic expansion for distributions of $C(\alpha)$ test statistics, in: S. Fienberg et al., eds., *Mathematical Statistics and Probability Theory. Lecture Notes in Statist. No. 2* (Springer, Berlin) pp. 63–96.
- Cohen, A., H.B. Sackrowitz and W.E. Strawderman (1985), Multivariate locally most powerful unbiased tests, in: P.R. Krishnaiah, ed. *Multivariate Analysis VI* (North-Holland, New York) pp. 121–144.
- Efron, B. (1975), Defining the curvature of a statistical problem, *Ann. Statist.* **3**, 1189–1242.
- Ferguson, T.S. (1967), *Mathematical Statistics – A Decision Theoretic Approach* (Academic Press, New York).
- Giri, N. and J. Kiefer (1964), Local and asymptotic minimax properties of multivariate tests, *Ann. Math. Statist.* **39**, 21–35.
- Hayakawa, T. (1990), On tests for the mean direction of Langevin distribution, *Ann. Inst. Statist. Math.* **42**, 359–373.
- Isaacson, S.L. (1951), On the theory of unbiased tests of simple statistical hypothesis specifying the values of two or more parameters, *Ann. Math. Statist.* **22**, 217–234.
- John, S. (1971), Some optimal multivariate tests, *Biometrika* **58**, 123–127.
- Kariya, T. (1978), The general MANOVA problem, *Ann. Math. Statist.* **6**, 200–214.
- Kiefer, J. and R. Schwartz (1965), Admissible Bayes character of T^2 -, R^2 -, and other fully invariant tests for classical multivariate normal problems, *Ann. Math. Statist.* **36**, 747–770.
- Mardia, K.V. (1972), *Statistics of Directional Data* (Academic Press, London).
- Neyman, J. (1959), Optimal asymptotic tests of composite statistical hypotheses, in: *Probability and Statistics (The Harold Cramer Volume)* (Almqvist and Wiksells, Uppsala) pp. 213–234.
- Scheffé, H. (1959), *The Analysis of Variance* (Wiley, New York).
- SenGupta, A. and S.R. Jammalamadaka (1989), On the curvature, geometry and optimal testing for a circular normal distribution, Tech. Rept., Dept. of Statist., Univ. of California (Santa Barbara, CA).
- SenGupta, A. and L. Vermeire, (1986), Locally optimal tests for multiparameter hypotheses, *J. Amer. Statist. Assoc.* **81**, 819–825.
- Stein, C. (1955), A necessary and sufficient condition for admissibility, *Ann. Math. Statist.* **26**, 518–522.
- Tong, Y.L. (1980), *Probability Inequalities in Multivariate Distributions* (Academic Press, New York).